Backward Stochatic Differential Equations II

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Backward Stochastic Differential Equations on Manifolds II

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Abstract

In [1], we have studied a generalization of the problem of finding a martingale on a manifold whose terminal value is known. This article completes the results obtained in the first article by providing uniqueness and existence theorems in a general framework (in particular if positive curvatures are allowed), still using differential geometry tools.

1 Introduction

1.1 Setting of the problem

First, we introduce some notations and definitions. Unless otherwise stated, we shall work on a fixed finite time interval $[0;T]$; moreover, $(W_t)_{0\leq t\leq T}$ will always denote a Brownian Motion (BM for short) in $\mathbb{R}^{d_w}$, for a positive integer d_w. Moreover, Einstein's summation convention will be used for repeated indices in lower and upper position.

Let $(B_t^y)_{0\leq t\leq T}$ denote the $\mathbb{R}^d$-valued diffusion which is the unique strong solution of the following SDE :

$$\left\{\begin{array}{rcl} dB_t^y &=& b(B_t^y)dt+\sigma(B_t^y)dW_t \\ B_0^y &=& y, \end{array}\right. \tag{1.1}$$

where $\sigma\ :\mathbb{R}^d\to\mathbb{R}^{d\times d_w}$ and $b:\mathbb{R}^d\to\mathbb{R}^d$ are C^3 bounded functions with bounded partial derivatives of order 1, 2 and 3.

Let us recall the problem stated in [1]. We consider a manifold M endowed with a connection Γ, which defines an exponential mapping. On M, we study the uniqueness and existence of a solution to the equation (under infinitesimal form)

$$(M+D)_0 \begin{cases} X_{t+dt} = \exp_{X_t}(Z_t dW_t + f(B_t^y, X_t, Z_t)dt) \\ X_T = U \end{cases}$$

where $Z_t \in \mathcal{L}(\mathbb{R}^{d_w}, T_{X_t}M)$ and $f(B_t^y, X_t, Z_t) \in T_{X_t}M$.

When $f = 0$, this equation characterizes exactly martingales on M with terminal value U. They are the main tool to solve in a probabilistic way some PDEs such as the Dirichlet problem or the heat equation. Note in particular the link with variational problems : it is well-known that harmonic mappings (between manifolds) transform Brownian Motions into martingales (see for instance [6]); moreover, these mappings are critical points of the energy functional and can be used to model the state of equilibrium of liquid crystals (see the introduction of [5] for a brief discussion).
In the case of a non-vanishing drift term f, the solutions are more general processes which are linked to more general PDEs and mappings generalizing harmonic ones. One could see these mappings modelling the equilibrium state of a liquid crystal in an exterior field equal to the drift term f in equation $(M+D)_0$.
For more details about the links with PDEs, the reader is referred to Kendall ([7]) or Thalmaier ([12]); see also the introductions of [1], [10] and [11].

In local coordinates (x^i), the equation $(M+D)_0$ becomes the following backward stochastic differential equation (BSDE in short)

$$(M+D) \begin{cases} dX_t = Z_t dW_t + \left(-\frac{1}{2}\Gamma_{jk}(X_t)([Z_t]^k|[Z_t]^j) + f(B_t^y, X_t, Z_t)\right) dt \\ X_T = U. \end{cases}$$

We keep the same notations as in [1] : $(\cdot|\cdot)$ is the usual inner product in an Euclidean space, the summation convention is used, and $[A]^i$ denotes the i^{th} row of any matrix A; moreover,

$$\Gamma_{jk}(x) = \begin{pmatrix} \Gamma^1_{jk}(x) \\ \vdots \\ \Gamma^n_{jk}(x) \end{pmatrix} \tag{1.2}$$

is a vector in $\mathbb{R}^n$, whose components are the Christoffel symbols of the connection. We keep the notations Z_t for a matrix in $\mathbb{R}^{n\times d_w}$ and f for a mapping from $\mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R}^{n\times d_w}$ to $\mathbb{R}^n$. The process X will take its values in a compact set, and a solution of equation $(M+D)_0$ will be a pair of processes (X, Z) in $M \times (\mathbb{R}^{d_w} \otimes TM)$ such that X is continuous and $\mathbb{E}(\int_0^T \|Z_t\|_r^2 dt) < \infty$ for a Riemannian norm $\|\cdot\|_r$; in global coordinates $O \subset \mathbb{R}^n$, $(X, Z) \in O \times \mathbb{R}^{n\times d_w}$ and $\mathbb{E}(\int_0^T \|Z_t\|^2 dt) < \infty$ (see below for the definitions of the norms). We gave in [1] existence and uniqueness results for the solutions of the BSDE $(M+D)$ in two different frameworks : firstly when the drift f did not depend on the process (Z_t), and secondly for a "general" f (i.e. depending on the process (Z_t)), but only for the Levi-Civita connection and

in nonpositive curvatures.
In this article we extend these results for "general" drifts f to other manifolds, without further hypothesis on the curvature as above. In this case, we need (unlike in [1]) to prove an exponential integrability condition like

$$\mathbb{E}\left(e^{\mu\int_0^T \|Z_s\|^2 ds} < \infty\right).$$

This allows the construction of a submartingale, which is the crux of the matter, but this leads to calculus which is much more intricate than in [1]. More precisely, we study two cases : on the one hand, the case of a general connection (i.e. not depending on the Riemannian structure defined on the manifold M) for which we give only results on small domains; on the other hand, the case of a manifold endowed with its Levi-Civita connection and whose sectional curvatures are allowed to be positive.

In Section 2, we prove some estimates and technical lemmas which are useful in Section 3, devoted to prove uniqueness results. Section 4 deals with existence results; it is very similar to Section 4 of [1], so we give the main results generally without proof, except when the arguments are more complicated than in [1]. To end the article, in Section 5 we recall briefly the links to the Dirichlet problem.

1.2 Notations and hypothesis

In all the article, we suppose that a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{0\leq t\leq T})$ (verifying the usual conditions) is given (with $T < \infty$ a deterministic time) on which $(W_t)_t$ denotes a d_w-dimensional BM. Moreover, we always deal with a complete Riemannian manifold M of dimension n, endowed with a linear symmetric (i.e. torsion-free) connection whose Christoffel symbols Γ^i_{jk} are smooth; the connection does not depend *a priori* on the Riemannian structure.

On M, δ denotes the Riemannian distance; $|u|_r$ is the Riemannian norm for a tangent vector u and $|u'|$ the Euclidean norm for a vector u' in $\mathbb{R}^n$. If h is a smooth real function defined on M, its differential is denoted by Dh or h'; the Hessian Hess $h(x)$ is a bilinear form the value of which is denoted by Hess $h(x) < u, \overline{u} >$, for tangent vectors (at x) u and $\overline{u}$.
For $\beta \in \mathbb{N}^*$, we say that a function is C^β on a closed set F if it is C^β on an open set containing F. Recall also that a real function χ defined on M is said to be convex if for any M-valued geodesic γ, $\chi \circ \gamma$ is convex in the usual sense (if χ is smooth, this is equivalent to require that Hess χ be nonnegative).
For a matrix z with n rows and d columns, ${}^t z$ denotes its transpose,

$$\|z\| = \sqrt{\mathrm{Tr}(z\,{}^t z)} = \sqrt{\sum_{i=1}^{d} |[{}^t z]^i|^2}$$

(Tr is the trace of a square matrix) and $\|z\|_r = \sqrt{\sum_{i=1}^{d} |[{}^t z]^i|_r^2}$ where the columns of z are considered as tangent vectors. The notation $\Psi(x, x') \approx \delta(x, x')^\nu$ means that there is a constant $c > 0$ such that

$$\forall x, x',\ \frac{1}{c}\ \delta(x,x')^\nu \le \Psi(x,x') \le c\ \delta(x,x')^\nu.$$

Before the general framework, let us give some additional notations which are specific to the Levi-Civita connection. In this case, we always assume that the injectivity radius R of M is positive and that its sectional curvatures are bounded above; we let K be the smallest nonnegative number dominating all the sectional curvatures, or 0 if they are all nonpositive. Finally we recall from [6] the definition of a regular geodesic ball.

Definition 1.2.1 *A closed geodesic ball $\mathcal{B}$ of radius ρ and center p is said to be regular if*
(i) $\rho\sqrt{K} < \frac{\pi}{2}$
(ii) The cut locus of p does not meet $\mathcal{B}$.

For an introductory course in Riemannian geometry, the reader is referred to Boothby ([2]) and for further facts about curvature, to Lee ([8]).

Let us come back to general connections. Throughout the article, we consider an open set O of M, relatively compact in a local chart and an open set $\omega \neq \emptyset$ relatively compact in O, verifying that

• There is a unique geodesic in $\overline{O}$, linking any two points of $\overline{O}$, and depending smoothly on its endpoints;
• $\overline{\omega} = \{\chi \le c\}$, the sublevel set of a smooth convex function χ defined on O.
Note that O will be as well considered as a subset of $\mathbb{R}^n$. In the case of a general connection, it is well-known that any point x of M has a neighbourhood O for which the first property holds; and when the Levi-Civita connection is used, the first property is also true for a regular geodesic ball (see Theorem (1.7) in [6]).
Finally we always assume two hypotheses on f :

$$\exists L > 0,\ \forall b, b' \in \mathbb{R}^d, \forall (x,z) \in O \times \mathcal{L}(\mathbb{R}^{d_w}, T_x M), \forall (x',z') \in O \times \mathcal{L}(\mathbb{R}^{d_w}, T_{x'} M),$$

$$\left| \overset{x'}{\underset{x}{\|}} f(b,x,z) - f(b',x',z') \right|_r \le L \Bigg((|b-b'| \ + \ \delta(x,x'))(1 + \|z\|_r + \|z'\|_r) + \ \left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r \Bigg) \tag{1.3}$$

and

$$\exists L_2 > 0, \exists x_0 \in O, \forall b \in \mathbb{R}^d, |f(b,x_0,0)|_r \le L_2. \tag{1.4}$$

The first one is a "geometrical" Lipschitz condition on f; this special form is needed to get an expression which is invariant under changes of coordinates. The second one means that f is bounded with respect to the first argument. Finally we denote by

$$\overset{x'}{\underset{x}{\|}} z$$

the parallel transport (defined by the connection) of the d_w columns of the matrix z (considered as tangent vectors) along the unique geodesic between x and x'.

Remark. In fact, condition (1.3) can be weakened by splitting it into two conditions : yet a Lipschitz condition on b and z

$$|f(b,x,z) - f(b',x,z')|_r \leq L\Bigg(|b-b'|(1+\|z\|_r + \|z'\|_r) + \|z-z'\|_r \Bigg)$$

and the following "monotonicity" condition on x

$$D\Psi \cdot \left(\begin{array}{c} f(b,x,z) \\ f(b,x', \overset{x'}{\underset{x}{\|}} z) \end{array} \right) \geq \mu \Psi(x,x')(1+\|z\|)$$

for a real constant μ independent of b, x, z (where Ψ is replaced by δ in the case of the Levi-Civita connection). This "monotonicity" condition replaces here the well-known monotonicity condition involving the inner product in an Euclidean space (see e.g. Assumption (4) in [3] or Assumption $(H3)$ in [9]). As in these references, we need also some additional conditions on f, in particular continuity in the x variable.
Note that here we have a lower bound on $D\Psi$ (and not an upper bound as in the articles cited above) because in the equation $(M+D)$, the drift f is given with a "plus" sign.

The proof of uniqueness is similar to the one in the Lipschitz case; for the existence result, we approximate f by Lipschitz functions f_n and pass through the limit in equation $(M+D)$; it involves more intricate computations (in particular to keep the assumption that the functions f_n are pointing outward on the boundary of $\overline{\omega}$, see Subsection 1.3 below). Details will appear elsewhere.

To end this part, notice that the same letter C will often stand for different constant numbers.

1.3 The main result

Before performing calculations, we give the main theorem of the article. Let us first introduce a technical but natural hypothesis, which we will make precise in Section 4 :

(H) f is pointing outward on the boundary of $\overline{\omega}$.

Then, under the notations above, we can state :

Theorem 1.3.1 *We consider the BSDE* $(M+D)$ *with terminal random variable* $U \in \overline{\omega} = \{\chi \leq c\}$*. We suppose that* f *verifies conditions* (1.3), (1.4), *and that* χ *is strictly convex (i.e.* Hess χ *is positive definite).*

(i) Each point q *of* M *has a neighbourhood* $O_q \subset O$ *such that, if* $\overline{\omega} \subset O_q$ *and* f *verifies hypothesis* (H)*, then the BSDE* $(M+D)$ *has a unique solution* $(X_t, Z_t)_{0 \leq t \leq T}$ *such that* X *remains in* $\overline{\omega}$*. The neighbourhood* O_q *depends on the geometry of the manifold, but not on the constants* L *and* L_2 *defined in* (1.3) *and* (1.4).

(ii) If the Levi-Civita connection is used and $\overline{\omega} \subset \mathcal{B}$*, a regular geodesic ball, and if* f *verifies hypothesis* (H)*, then the BSDE* $(M+D)$ *has a unique solution* $(X_t, Z_t)_{0 \leq t \leq T}$ *with* X *in* $\overline{\omega}$*. Moreover, if* $\overline{\omega} = \mathcal{B}$*, the hypothesis on the strict convexity of* χ *is satisfied by taking* $\chi = \delta^2$*.*

Theorem 1.3.1 brings together Theorems 3.2.2 and 3.3.4 and the results of Section 4. In Theorem 5.2.1, we will extend this theorem to random time intervals $[0; \tau]$, for stopping times τ which verify an exponential integrability condition and "sufficiently small" drifts f.

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2 Preliminary estimates

We first recall elementary results about Itô's formula and parallel transport. Then we give some geometrical estimates for the distance function on $M \times M$. These results are nontrivial generalizations (mainly in the case of the Levi-Civita connection) of results of [4] and [10]. In this section, the covariant derivative of a vector field $z(t)$ along a curve γ_t is denoted by $\nabla_{\dot{\gamma}_t} z(t)$.

2.1 Itô's formula on manifolds

Consider two solutions (X^1, Z^1) and (X^2, Z^2) of equation $(M+D)$ with terminal values U^1 and U^2, such that X^1 and X^2 remain in O. Let

$$\tilde{X} = (X^1, X^2) \quad \text{and} \quad \tilde{Z} = \begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix};$$

then, for a smooth function Ψ defined on $O \times O$, Itô's formula is written

$$
\begin{aligned}
\Psi(\tilde{X}_t) - \Psi(\tilde{X}_0) &= \int_0^t D\Psi(\tilde{X}_s)\left(\tilde{Z}_s dW_s\right) \\
&\quad + \int_0^t D\Psi(\tilde{X}_s)\begin{pmatrix} f(B_s^y, X_s^1, Z_s^1) - \frac{1}{2}\Gamma_{jk}(X_s^1)([Z_s^1]^k|[Z_s^1]^j) \\ f(B_s^y, X_s^2, Z_s^2) - \frac{1}{2}\Gamma_{jk}(X_s^2)([Z_s^2]^k|[Z_s^2]^j) \end{pmatrix} ds \\
&\quad + \frac{1}{2}\int_0^t \mathrm{Tr}\left({}^t\tilde{Z}_s D^2\Psi(\tilde{X}_s)\tilde{Z}_s\right) ds \\
&= \int_0^t D\Psi(\tilde{X}_s)\left(\tilde{Z}_s dW_s\right) \\
&\quad + \frac{1}{2}\int_0^t \left(\sum_{i=1}^{d_w} {}^t[{}^t\tilde{Z}_s]^i \text{Hess } \Psi(\tilde{X}_s)[{}^t\tilde{Z}_s]^i\right) ds \\
&\quad + \int_0^t D\Psi(\tilde{X}_s)\begin{pmatrix} f(B_s^y, X_s^1, Z_s^1) \\ f(B_s^y, X_s^2, Z_s^2) \end{pmatrix} ds
\end{aligned} \tag{2.1}
$$

where $D^2\Psi$ is the second order derivatives matrix (remember also notation (1.2) and that $[{}^t\tilde{Z}_s]^i$ denotes the i^{th} column of the matrix $\tilde{Z}_s$; it is a vector in $\mathbb{R}^{2n}$). Moreover, for a smooth function h on O and a solution (X, Z) of $(M + D)$, we get a similar formula, replacing $\tilde{X}$ by X and $\tilde{Z}$ by Z.

2.2 A comparison result about parallel transports

The relatively compact set O is considered here as a subset of $\mathbb{R}^n$; the following proposition gives in O a comparison result between the parallel transports defined by the Euclidean structure and the connection.

Proposition 2.2.1 *There is a $C > 0$ such that for every $(x, x') \in O \times O$ and $(z, z') \in T_xM \times T_{x'}M$, we have*

$$
\left|\overset{x'}{\underset{x}{\|}} z - z'\right|_r \leq C\left(|z - z'| + \delta(x, x')(|z| + |z'|)\right)
$$

and

$$
|z - z'| \leq C\left(\left|\overset{x'}{\underset{x}{\|}} z - z'\right|_r + \delta(x, x')(|z|_r + |z'|_r)\right). \tag{2.2}
$$

Proof. The case of the Levi-Civita connection has been treated in [1]. We give the proof here in the case of a general connection, not depending *a priori* on the Riemannian structure. Using the triangle inequality and the equivalence of the Euclidean and Riemannian norms on compact domains, we first remark that it is sufficient to prove the existence of $C > 0$ such that

$$
\forall (x, x') \in O, \forall z \in \mathbb{R}^n, \left|\overset{x'}{\underset{x}{\|}} z - z\right| \leq C\delta(x, x')|z|. \tag{2.3}
$$

In fact, using the linearity property of the parallel transport, it is sufficient to prove (2.3) for $|z| = 1$. Define

$$\begin{array}{rcl} \tau : \overline{O} \times \overline{O} \times S(0;1) & \to & \overline{O} \\ (x, x', z) & \mapsto & \overset{x'}{\underset{x}{\|}} z \end{array}$$

where $S(0;1)$ is the sphere of radius 1 in $\mathbb{R}^n$. It is a smooth mapping. Indeed, let γ be the geodesic such that $\gamma(0) = x$ and $\gamma(1) = x'$; we have supposed that γ depends smoothly on its endpoints x and x'. Moreover, if we note

$$\forall t \in [0;1], \quad z(t) = \overset{\gamma_t}{\underset{x}{\|}} z,$$

the equation of the parallel transport $\nabla_{\dot{\gamma}_t} z(t) = 0$ is written in local coordinates

$$\forall k, \quad \dot{z}^k(t) + \Gamma^k_{jl}(\gamma_t)\dot{\gamma}^j_t z^l(t) = 0.$$

The general theory of differential equations gives then the smoothness of τ. Now since (x, x', z) is in a compact set, we get, for a constant C independent of x, x' and z

$$|\tau(x, x', z) - \tau(x, x, z)| \leq C\delta(x, x').$$

This is exactly (2.3) for z such that $|z| = 1$. This completes the proof. □

As a consequence, on the relatively compact set $O \subset \mathbb{R}^n$, (1.3) becomes

$$\exists L' > 0, \ \forall b, b' \in \mathbb{R}^d, \ \forall (x, z), (x', z') \in O \times \mathbb{R}^{n \times d_w},$$
$$|f(b, x, z) - f(b', x', z')| \leq L'((|b - b'| + |x - x'|)(1 + \|z\| + \|z'\|) + \|z - z'\|).$$

2.3 A local lower bound on a Hessian

In this paragraph, we suppose that there exists a nonnegative, smooth and convex function Ψ on the product $\overline{\omega} \times \overline{\omega}$ (i.e. convex on an open set containing this set) which vanishes only on the diagonal $\Delta = \{(x, x)/x \in \overline{\omega}\}$ ($\overline{\omega}$ is said to have Γ-convex geometry); besides, we suppose that $\Psi \approx \delta^p$ for a $p \geq 2$. Note that since Ψ is smooth, p is an even integer (see Remark 2 after the proof of Lemma 2.3.1.

Let (x, x') be a point in $\overline{\omega} \times \overline{\omega}$. For notational convenience, we keep the same notation $\overline{\omega} \times \overline{\omega}$ for the image of this compact set in the local coordinates considered below, and we write f for $f(b, x, z)$ and f' for $f(b, x', z')$ (note that we use here the same b). Take a local chart (ϕ, ϕ) in which (x, x') has coordinates $(\hat{x}, \hat{x}')$; if $(\partial_1, \ldots, \partial_{2n})$ denotes the natural dual basis of these coordinates, then

$$\begin{pmatrix} f \\ f' \end{pmatrix} = \sum_{i=1}^{n} \left(f^i \partial_i + f'^i \partial_{i+n} \right) \tag{2.4}$$

and

$$(\text{Hess } \Psi(x, x'))_{ij} = \partial_{ij}\Psi(x, x') - \overline{\Gamma}^k_{ij}(x, x')\partial_k \Psi(x, x')$$

where the $\overline{\Gamma}_{ij}^k$ are the Christoffel symbols associated with the product connection. Recall that, if we note $\overline{\Gamma}_{ij} = (\overline{\Gamma}_{ij}^1, \dots, \overline{\Gamma}_{ij}^{2n})$, they are given by

$$\overline{\Gamma}_{ij}(x,x') = \begin{cases} (\Gamma_{ij}^1(x), \dots, \Gamma_{ij}^n(x), 0, \dots, 0) & \text{if } \ i,j \leq n, \\ (0, \dots, 0, \Gamma_{i-n,j-n}^1(x'), \dots, \Gamma_{i-n,j-n}^n(x')) & \text{if } \ i,j > n, \\ (0, \dots 0, 0, \dots, 0) & \text{if } \ i \leq n < j \ \text{ or } \ j \leq n < i. \end{cases}$$

We put

$$\begin{pmatrix} A & E \\ {}^tE & B \end{pmatrix}_{ij} := \partial_{ij}\Psi - \overline{\Gamma}_{ij}^k \partial_k \Psi$$

where A, B and E are square matrices of size n.

Let $v = (v_1, \dots, v_{2n}) = (\hat{x} - \hat{x}', \hat{x}')$ be new coordinates and $(\tilde{\partial}_1, \dots, \tilde{\partial}_{2n})$ be the natural dual basis. Then the diagonal Δ is given by the equation $\{v_1 = \cdots = v_n = 0\}$ and for $i = 1, \dots, n$, $\tilde{\partial}_i = \partial_i$ and $\tilde{\partial}_{i+n} = \partial_i + \partial_{i+n}$. Therefore in these new coordinates,

$$\begin{pmatrix} z \\ z' \end{pmatrix} = \sum_{i=1}^{n} \Big((z^i - z'^i)\tilde{\partial}_i + z'^i \tilde{\partial}_{i+n} \Big)$$

and

$${}^t\begin{pmatrix} z \\ z' \end{pmatrix} \text{Hess } \Psi(x,x') \begin{pmatrix} z \\ z' \end{pmatrix} = {}^t\begin{pmatrix} z - z' \\ z' \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{E} \\ {}^t\tilde{E} & \tilde{B} \end{pmatrix} \begin{pmatrix} z - z' \\ z' \end{pmatrix},$$

where the square matrices $\tilde{A}, \tilde{B}$ and $\tilde{E}$ are given by

$$\forall i,j = 1, \dots, n, \begin{cases} \tilde{A}_{ij} = \tilde{\partial}_{ij}\Psi - \sum_{k=1}^n \overline{\Gamma}_{ij}^k \tilde{\partial}_k \Psi \\ \tilde{B}_{ij} = \tilde{\partial}_{i+n,j+n}\Psi - \sum_{k=n+1}^{2n} \overline{\Gamma}_{i+n,j+n}^k \tilde{\partial}_k \Psi \\ \tilde{E}_{ij} = \tilde{\partial}_{i,j+n}\Psi - \sum_{k=1}^{2n} \overline{\Gamma}_{i,j+n}^k \tilde{\partial}_k \Psi = \tilde{\partial}_{i,j+n}\Psi. \end{cases}$$

Lemma 2.3.1 *In v-coordinates, we have the following estimates :*
(i) If $i \leq n$, $\tilde{\partial}_i \Psi(x,x') \leq C\delta^{p-1}(x,x')$.
(ii) If $i > n$, $\tilde{\partial}_i \Psi(x,x') \leq C\delta^{p}(x,x')$.

Proof. Let V denote (x,x') in v-coordinates and $\hat{p}(V)$ the projection of a vector V onto Δ : if $V = (v_1, \dots, v_{2n})$, then $\hat{p}(V) = (0, \dots, 0, v_{n+1}, \dots, v_{2n})$. Remark first that $|V - \hat{p}(V)| \approx \delta(x,x')$; then use a Taylor expansion of order p of Ψ near the diagonal (remember that $\Psi \approx \delta^p$) :

$$\begin{aligned} \Psi(V) &= \frac{1}{p!} \sum_{1 \leq i_1, \dots, i_p \leq n} \tilde{\partial}_{i_1 \dots i_p} \Psi(\hat{p}(V))(V - \hat{p}(V))_{i_1} \dots (V - \hat{p}(V))_{i_p} \\ &+ \int_0^1 \frac{(1-t)^p}{p!} \sum_{1 \leq j_1, \dots, j_{p+1} \leq n} \tilde{\partial}_{j_1 \dots j_{p+1}} \Psi(V_t)(V - \hat{p}(V))_{j_1} \dots (V - \hat{p}(V))_{j_{p+1}} dt \end{aligned}$$

where $V_t = \hat{p}(V) + t(V - \hat{p}(V))$. By differentiating this equality, we get the two estimates (note that if $i \leq n$, $\tilde{\partial}_i$ corresponds to the differentiation with respect to the i^{th} term of $V - \hat{p}(V)$, and if $i > n$, $\tilde{\partial}_i$ corresponds to the differentiation with respect to the i^{th} term of $\hat{p}(V)$). $\square$

Remark 1 : By iterating the procedure, one can show in particular that there is a constant $C > 0$ such that for x and x' in $\overline{O}$, $|\tilde{E}_{ij}(x, x')| \leq C\delta(x, x')^{p-1}$.

Remark 2 : A consequence of the Taylor expansion in the proof above is that p is an even integer. Indeed, it is straightforward that p is an integer, and if we change V by $-V + 2\hat{p}(V)$, we conclude that p cannot be odd since Ψ is nonnegative.

We prove now a lower bound on Hess Ψ.

Proposition 2.3.2 *Suppose that there is a neighbourhood O_Δ of the diagonal in $\overline{\omega}$ and a $\eta > 0$ such that, for all x, x' in O_Δ and z in $\mathbb{R}^n$*

$$^t z\tilde{A}z \geq \eta\delta^{p-2}(x, x')|z|^2. \tag{2.5}$$

Then there are positive constants α and β such that

$$\forall (x, x') \in O_\Delta, \forall (z, z') \in T_x M \times T_{x'} M,$$

$$\begin{aligned} {}^t\begin{pmatrix} z \\ z' \end{pmatrix} \text{Hess } \Psi(x, x') \begin{pmatrix} z \\ z' \end{pmatrix} &\geq \alpha\delta^{p-2}(x, x') \left| \overset{x'}{\underset{x}{\|}} z - z' \right|_r^2 \\ &\quad -\beta\Psi(x, x')(|z|_r^2 + |z'|_r^2). \end{aligned} \tag{2.6}$$

Proof. We work in v-coordinates. Using the hypothesis on $\tilde{A}$ and the nonnegativity of $\tilde{B}$ (since Hess Ψ is nonnegative), we write

$$\begin{aligned} {}^t\begin{pmatrix} z \\ z' \end{pmatrix} \text{Hess } \Psi(x, x') \begin{pmatrix} z \\ z' \end{pmatrix} &= {}^t(z - z')\tilde{A}(z - z') + {}^t(z')\tilde{B}z' \\ &\quad + {}^t(z - z')\tilde{E}z' + {}^t(z')^t\tilde{E}(z - z') \\ &\geq \eta\delta^{p-2}(x, x')|z - z'|^2 - 2|{}^t(z - z')\tilde{E}z'|. \end{aligned}$$

But using the remark after Lemma 2.3.1, $\|\tilde{E}\| \leq C\delta^{p-1}$, so

$$\begin{aligned} |{}^t(z - z')\tilde{E}z'| &\leq C\delta^{p-1}(x, x')|z'|\,|z - z'| = C\delta^{p-2}(x, x')\,(\delta(x, x')|z'|)\,|z - z'| \\ &\leq \frac{\eta}{2}\delta^{p-2}(x, x')|z - z'|^2 + \zeta\Psi(x, x')|z'|^2 \end{aligned}$$

where we have used the classical inequality : $ab \leq \varepsilon a^2 + (1/\varepsilon)b^2$ for any $\varepsilon > 0$, and $\Psi \approx \delta^p$. This bound gives

$${}^t\begin{pmatrix} z \\ z' \end{pmatrix} \text{Hess } \Psi(x, x') \begin{pmatrix} z \\ z' \end{pmatrix} \geq \frac{\eta}{2}\delta^{p-2}(x, x')|z - z'|^2 - \zeta\Psi(x, x')|z'|^2.$$

Using this estimate, the equivalence of norms and Proposition 2.2.1 gives exactly the result. $\square$

An example of a function Ψ which verifies the hypothesis of Proposition 2.3.2 is the convex function constructed by Emery (see Lemma (4.59) in [4]); in v-coordinates as defined above, it can be written

$$\Psi(v_1,\ldots,v_n,v_{n+1},\ldots,v_{2n}) = \frac{1}{2}\left(\varepsilon^2 + \sum_{k=n+1}^{2n} v_k^2\right)\left(\sum_{k=1}^{n} v_k^2\right). \tag{2.7}$$

Emery has shown that, for ε sufficiently small, Ψ is convex near the diagonal. Moreover, it also verifies near the diagonal the estimate (2.5). Indeed, with the notations above, we have for $i,j \le n$:

$$\tilde{A}_{ij}(v) = \tilde{\partial}_{ij}\Psi(v) - \sum_{k=1}^{n} \overline{\Gamma}_{ij}^{k}(v)\tilde{\partial}_k\Psi(v).$$

An explicit calculation shows that

$$|\tilde{\partial}_k\Psi(v)| \le C\left(\sum_{k=1}^{n} v_k^2\right)^{\frac{1}{2}} \quad \text{and} \quad \tilde{\partial}_{ij}\Psi(v) = \begin{cases} (\varepsilon^2 + \sum_{k=n+1}^{2n} v_k^2) & \text{if } i=j \\ 0 & \text{otherwise.} \end{cases}$$

Then, in the sense of matrices, $(\tilde{\partial}_{ij}\Psi)_{ij} \ge \varepsilon^2 I_n$ (where I_n is the identity matrix of $\mathbb{R}^n$). But, since $(\sum_{k=1}^{n} v_k^2)^{\frac{1}{2}} \approx |V - p(V)| \approx \delta(x,x')$, the other matrix

$$(\sum_{k=1}^{n} \overline{\Gamma}_{ij}^{k}(v)\tilde{\partial}_k\Psi(v))_{ij}$$

is negligible compared to $(\tilde{\partial}_{ij}\Psi)_{ij}$, near the diagonal. Thus $(\tilde{A}_{ij}(v))_{ij} \ge (\varepsilon^2/2)I_n$ in a neighbourhood of Δ; this is exactly (2.5) since, in this example, $\Psi \approx \delta^2$ and $p=2$.

2.4 Estimates of the derivatives of the distance

In this subsection, the Levi-Civita connection is used. Then the geodesic distance $(x,x') \mapsto \delta(x,x')$ is defined on $M \times M$ and is smooth except on the cut locus and the diagonal $\{x = x'\}$. We want to estimate its first and second derivatives when $M \times M$ is endowed with the product Riemannian metric. If $\tilde{x} = (x,x')$ is a point which is not in the cut locus or the diagonal, there exists a unique minimizing geodesic $\gamma(t)$, $0 \le t \le 1$, from x to x'. If u_t is a vector of $T_{\gamma(t)}M$, we can decompose u_t as $v_t + w_t$, where v_t is the orthogonal projection of u_t on $\dot{\gamma}(t)$; the vectors v_t and w_t are respectively called the tangential and orthogonal components of u_t. If $u = (u_0,u_1)$ is a vector of $T_{\tilde{x}}(M \times M)$, (v_0,v_1) and (w_0,w_1) are also called its tangential and orthogonal components. In the sequel, we put

$$y := \sqrt{K}\frac{\delta(\tilde{x})}{2}.$$

We start with a technical lemma.

Lemma 2.4.1 *For* $0 < t < \pi$*, let* $h(t) := \frac{\sin t}{t}$ *and*

$$H(t, \beta) := \frac{1 - h(t)\cos(t + 2\beta)}{\sin^2 \beta + \sin^2(t + \beta)};$$

then, for $0 < y < \frac{\pi}{2}$*, we have*

$$\max_{0 \leq \beta \leq \pi - 2y} (H(2y, \beta)) = \frac{1 + h(2y)}{1 + \cos(2y)}.$$

Proof. Let $D(t, \beta) := \sin^2 \beta + \sin^2(t + \beta)$ be the denominator of H; then some simple trigonometry gives

$$\begin{aligned} D(t, \beta) &= \frac{1}{2}(1 - \cos(2\beta)) + \frac{1}{2}(1 - \cos(2t + 2\beta)) \\ &= 1 - \cos t \cdot \cos(t + 2\beta). \end{aligned}$$

When we differentiate H with respect to β, we get

$$\frac{D(t, \beta)(2h(t)\sin(t + 2\beta)) - (1 - h(t)\cos(t + 2\beta))(2\cos t \cdot \sin(t + 2\beta))}{D(t, \beta)^2};$$

therefore, we have

$$\frac{\partial H}{\partial \beta}(t, \beta) = 2(h(t) - \cos t)\frac{\sin(t + 2\beta)}{D(t, \beta)^2}.$$

In particular, since $h(t) \geq \cos t$ on $[0; \pi]$,

$$sign(\frac{\partial H}{\partial \beta}(t, \beta)) = sign(\sin(t + 2\beta))$$

and the function $H(2y, \cdot)$ reaches its maximum (for $0 \leq \beta \leq \pi - 2y$) at $\beta_m = \frac{\pi}{2} - y$. A simple calculation gives

$$H(2y, \beta_m) = \frac{1 + h(2y)}{1 + \cos(2y)}.$$

This completes the proof. □

Now we give the estimates on the first two derivatives of the distance.

Lemma 2.4.2 *Let* $\tilde{x}$ *be a point of* $M \times M$ *such that* $0 < \delta(\tilde{x}) < \pi/\sqrt{K}$*. Let* $u = (u_0, u_1)$ *be a vector of* $T_{\tilde{x}}(M \times M)$ *and let* v *and* w *be its tangential and orthogonal components. Then*

$$|\delta'(\tilde{x}) < u > | = \left| \overset{x'}{\underset{x}{\|}} v_0 - v_1 \right|_r \tag{2.8}$$

and

$$\text{Hess } \delta(\tilde{x}) < u, u > \geq \frac{1}{\delta(\tilde{x})} \left| \overset{x'}{\underset{x}{\|}} w_0 - w_1 \right|_r^2 - \frac{K}{2}\delta(\tilde{x}) \frac{1 + \frac{\sin(\sqrt{K}\delta(\tilde{x}))}{\sqrt{K}\delta(\tilde{x})}}{1 + \cos(\sqrt{K}\delta(\tilde{x}))} |w|_r^2. \tag{2.9}$$

Proof. The equality (2.8) is proved in [1]. Now let $J_w(t)$ be the normal Jacobi field along $\gamma(t)$ satisfying $J_w(0) = w_0$ and $J_w(1) = w_1$. From (1.1.7) of [10] and the proof of Lemma 2.3.1 of [1], we have

$$\begin{aligned}\text{Hess }\delta(\tilde{x}) < u, u > &\geq \frac{1}{\delta(\tilde{x})}\int_0^1 |\nabla_{\dot{\gamma}(t)} J_w(t)|_r^2 dt - K\delta(\tilde{x})\int_0^1 |J_w(t)|_r^2 dt \\ &\geq \frac{1}{\delta(\tilde{x})}\left|\overset{x'}{\underset{x}{\|}} w_0 - w_1\right|_r^2 - K\delta(\tilde{x})\int_0^1 |J_w(t)|_r^2 dt.\end{aligned}$$

The problem is to estimate $\int_0^1 |J_w(t)|_r^2 dt$. We know from [10] that

$$\frac{d^2}{dt^2}|J_w(t)|_r \geq -K\delta^2(\tilde{x})|J_w(t)|_r$$

at points t such that $J_w(t) \neq 0$. By comparing with the solution of the corresponding second order differential equation j, with $j(0) = |J_w(0)|_r$ and $j(1) = |J_w(1)|_r$, we have (recall that $y = \sqrt{K}\delta(\tilde{x})/2$) :

$$|J_w(t)|_r \leq j(t) = \alpha \sin(2yt + \beta) \text{ on } [0;1] ,$$

where $\alpha \geq 0$ and $0 \leq \beta \leq \pi - 2y$ are defined by

$$\alpha \sin\beta = |w_0|_r, \quad \alpha\sin(2y + \beta) = |w_1|_r.$$

In particular,

$$\alpha^2 = \frac{|w|_r^2}{\sin^2\beta + \sin^2(2y+\beta)}.$$

Now note that

$$\begin{aligned}\int_0^1 \sin^2(2yt+\beta)dt &= \frac{1}{2}\left[t - \frac{\sin(4yt+2\beta)}{4y}\right]_0^1 \\ &= \frac{1}{8y}(4y + \sin(2\beta) - \sin(4y+2\beta)) \\ &= \frac{1}{8y}(4y - 2\cos(2y+2\beta)\sin(2y)) \\ &= \frac{1}{2} - \frac{1}{2}h(2y)\cos(2y+2\beta).\end{aligned}$$

Then

$$\int_0^1 |J_w(t)|_r^2 dt \leq \alpha^2 \int_0^1 \sin^2(2yt+\beta)dt = \frac{1 - h(2y)\cos(2y+2\beta)}{\sin^2\beta + \sin^2(2y+\beta)} \cdot \frac{|w|_r^2}{2}.$$

According to Lemma 2.4.1, the maximum, for $0 \leq \beta \leq \pi - 2y$, of the first ratio is

$$\frac{1 + h(2y)}{1 + \cos(2y)}.$$

Finally,

$$\text{Hess }\delta(\tilde{x}) < u,u >\geq \frac{1}{\delta(\tilde{x})}\left|\overset{x'}{\underset{x}{\|}} w_0 - w_1\right|_r^2 - \frac{K}{2}\delta(\tilde{x})\frac{1+h(2y)}{1+\cos(2y)}|w|_r^2.$$

This is (2.9). □

For a such that $1 < a < 2$, we introduce the function Ψ_a defined by

$$\Psi_a(x,x') = \sin^a\left(\sqrt{K}\frac{\delta(x,x')}{2}\right) = \sin^a y. \tag{2.10}$$

This is the function we will use to construct the submartingale in the uniqueness part, when the Levi-Civita connection is used. In view of Itô's formula and the uniqueness part, we want to obtain estimates on Hess Ψ_a.

Lemma 2.4.3 *We have the two following bounds for* Hess Ψ_a *:*
(i) Let $\beta > 1$. There is a constant $\alpha > 0$, and a neighbourhood V_β of the diagonal, depending on α, such that for any $\tilde{x} \in (V_\beta \setminus \Delta) \cap (\overline{\omega} \times \overline{\omega})$,

$$\text{Hess }\Psi_a(\tilde{x}) < u,u >\geq \alpha \sin^{a-2}(y)\left|\overset{x'}{\underset{x}{\|}} u_0 - u_1\right|_r^2 - a\beta\frac{K}{2}\Psi_a(\tilde{x})|w|_r^2. \tag{2.11}$$

(ii) For any $\tilde{x}$ such that $0 < \delta(\tilde{x}) < \pi/\sqrt{K}$ and any $u = (u_0, u_1) \in T_xM \times T_{x'}M$,

$$\text{Hess }\Psi_a(\tilde{x}) < u,u >\geq -a\frac{K}{2}\Psi_a(\tilde{x})|u|_r^2. \tag{2.12}$$

Proof. (i) We can write Hess Ψ_a explicitly :

$$\begin{aligned}\text{Hess }\Psi_a(\tilde{x}) < u,u > &= \frac{K}{4}a\sin^{a-2}y\left((a-1)\cos^2 y - \sin^2 y\right)(\delta'(\tilde{x}) < u >)^2 \\ &\quad + a\frac{\sqrt{K}}{2}\sin^{a-1}y\cos y \text{ Hess }\delta(\tilde{x}) < u,u > . \end{aligned}\tag{2.13}$$

Let us call, in the right part of (2.13), T_1 the first term and T_2 the second one. We want to bound below these two terms near the diagonal. For T_1, note that there is a neighbourhood V_1 of Δ, in $\overline{\omega} \times \overline{\omega}$, such that, for $\tilde{x} \in V_1$ (recall that $y = \sqrt{K}\frac{\delta(\tilde{x})}{2}$),

$$(a-1)\cos^2 y - \sin^2 y \geq \frac{a-1}{2} \quad\text{and}\quad \sin^{a-2} y \geq 1;$$

then using (2.8),

$$T_1 \geq \frac{K}{8}a(a-1)\sin^{a-2}y\left|\overset{x'}{\underset{x}{\|}} v_0 - v_1\right|_r^2. \tag{2.14}$$

For T_2, we use (2.9) to get the inequality

$$T_2 \geq a\frac{K}{4}\sin^{a-2}y\cos y\frac{\sin y}{y}\left|\overset{x'}{\underset{x}{\|}} w_0 - w_1\right|_r^2 - a\frac{K}{2}\Psi_a(\tilde{x})y\cot y\frac{1+h(2y)}{1+\cos(2y)}|w|_r^2.$$

There is again a neighbourhood V_2 of Δ, in $\overline{\omega} \times \overline{\omega}$, such that, for $\tilde{x} \in V_2$

$$\cos y \frac{\sin y}{y} \geq \frac{1}{2} \quad \text{and} \quad y \cot y \frac{1+h(2y)}{1+\cos(2y)} \leq \beta;$$

therefore, for $\tilde{x} \in V_2$,

$$T_2 \geq a\frac{K}{8} \sin^{a-2} y \left| \overset{x'}{\underset{x}{\|}} w_0 - w_1 \right|_r^2 - a\beta \frac{K}{2} \Psi_a(\tilde{x}) |w|_r^2. \tag{2.15}$$

Now (2.13) together with (2.14) and (2.15) imply the result.

(ii) The equality (2.13) gives

$$\begin{aligned} \text{Hess } \Psi_a(\tilde{x}) < u, u > \quad \geq \quad & -a\frac{K}{4} \Psi_a(\tilde{x}) (\delta'(\tilde{x}) < u >)^2 \\ & + a\frac{\sqrt{K}}{2} \sin^{a-1} y \cos y \text{ Hess } \delta(\tilde{x}) < u, u > . \end{aligned}$$

Now by (2.8), $(\delta'(\tilde{x}) < u >)^2 \leq 2|v|_r^2$; moreover, we know from estimate (1.1.2) from [10] that

$$\text{Hess } \delta(\tilde{x}) < u, u > \geq -\sqrt{K} \tan(y) |w|_r^2$$

so

$$\text{Hess } \Psi_a(\tilde{x}) < u, u > \geq -a\frac{K}{2} \Psi_a(\tilde{x}) |v|_r^2 - a\frac{K}{2} \Psi_a(\tilde{x}) |w|_r^2.$$

This is (2.12). □

3 Uniqueness

3.1 The general method

Consider two solutions $(X_t, Z_t)_{0 \leq t \leq T}$ and $(X'_t, Z'_t)_{0 \leq t \leq T}$ of $(M+D)$ such that X and X' remain in $\overline{\omega}$ and $X_T = Y_T = U$ (we will say sometimes "$\overline{\omega}$-valued solutions of $(M+D)$"). Let

$$\tilde{X}_s = (X_s, X'_s) \quad \text{and } \tilde{Z}_s = \begin{pmatrix} Z_s \\ Z'_s \end{pmatrix}.$$

To prove uniqueness, the idea is to show that the process $(S_t)_t = (\exp(A_t)\Psi(\tilde{X}_t))_t$ where

$$A_t = \lambda t + \mu \int_0^t (\|Z_s\|_r^2 + \|Z'_s\|_r^2) ds,$$

is a submartingale for appropriate nonnegative constants λ and μ, and a suitable function Ψ, smooth on $\overline{O}$. But to define such a process, we need to consider solutions verifying an integrability condition. This leads to the following definition.

Definition 3.1.1 *If α is a positive constant, let $(\mathcal{E}_\alpha)$ be the set of $\overline{\omega}$-valued solutions of $(M+D)$ satisfying*

$$\mathbb{E}\exp\left(\alpha\int_0^T \|Z_s\|_r^2 ds\right) < \infty.$$

Actually, we will see that for α small, $(\mathcal{E}_\alpha)$ contains any solution of the equation $(M+D)$; we will use Lemma 3.4.2 of [1] which we recall here and which is valid for a general connection.

Lemma 3.1.2 *Suppose that we are given a positive constant α and a C^2 function ϕ on $\overline{\omega}$ satisfying $C_{min} \leq \phi(x) \leq C_{max}$ for some positive C_{min} and C_{max}. Suppose moreover that* Hess $\phi + 2\alpha\phi \leq 0$ *on $\overline{\omega}$; this means that*

$$\text{Hess }\phi(x) < u, u > +2\alpha\phi(x)|u|_r^2 \leq 0. \tag{3.1}$$

Then, for every $\varepsilon > 0$, any $\overline{\omega}$-valued solution of $(M+D)$ belongs to $(\mathcal{E}_{\alpha-\varepsilon})$.

Once we get the integrability property, it remains to show that $(S_t)_t$ is indeed a submartingale. We now turn to that problem. An application of Itô's formula (2.1) gives

$$\begin{aligned}
S_t - S_0 &= \int_0^t e^{A_s} d(\Psi(\tilde{X}_s)) + \int_0^t e^{A_s}(\lambda + \mu(\|Z_s\|_r^2 + \|Z'_s\|_r^2))\Psi(\tilde{X}_s) ds \\
&= \int_0^t e^{A_s} D\Psi(\tilde{X}_s)\left(\tilde{Z}_s dW_s\right) \\
&\quad + \frac{1}{2}\int_0^t e^{A_s}\left(\sum_{i=1}^{d_w} {}^t[{}^t\tilde{Z}_s]^i \text{Hess } \Psi(\tilde{X}_s)[{}^t\tilde{Z}_s]^i\right) ds \\
&\quad + \int_0^t e^{A_s} D\Psi(\tilde{X}_s)\begin{pmatrix} f(B_s^y, X_s, Z_s) \\ f(B_s^y, X'_s, Z'_s)\end{pmatrix} ds \\
&\quad + \int_0^t e^{A_s}\Psi(\tilde{X}_s)(\lambda + \mu(\|Z_s\|_r^2 + \|Z'_s\|_r^2)) ds.
\end{aligned} \tag{3.2}$$

It is clear that the submartingale property will hold if we show the nonnegativity of the sum

$$\begin{aligned}
\frac{1}{2}\sum_{i=1}^{d_w} {}^t[{}^t\tilde{Z}_t]^i \text{Hess } \Psi(\tilde{X}_t)[{}^t\tilde{Z}_t]^i \quad &+ \quad D\Psi(\tilde{X}_t)\begin{pmatrix} f(B_t^y, X_t, Z_t) \\ f(B_t^y, X'_t, Z'_t)\end{pmatrix} \\
&+ \quad (\lambda + \mu(\|Z_t\|_r^2 + \|Z'_t\|_r^2))\Psi(\tilde{X}_t).
\end{aligned} \tag{3.3}$$

Before giving the details of the proof, we first recall the following upper bound on the second term in (3.3).

Lemma 3.1.3 *Suppose that Ψ is a smooth nonnegative function on $\overline{\omega}\times\overline{\omega}$, vanishing only on the diagonal and such that $\Psi \approx \delta^p$ (we have seen then that p is an even* *positive integer). Then, in local coordinates, there is $C > 0$ such that, for all x, x' in $\overline{\omega}$, z, z' in $\mathbb{R}^{n\times d_w}$ and $b \in \mathbb{R}^d$*

$$\begin{aligned}
\left| D\Psi \cdot \begin{pmatrix} f(b,x,z) \\ f(b,x',z') \end{pmatrix} \right| &\leq C\delta^{p-1}(x,x')\left(\delta(x,x')(1+\|z\|+\|z'\|)+\|z-z'\|\right) \\
&\leq C_\varepsilon \Psi(x,x')(1+\|z\|_r+\|z'\|_r) \\
&\quad +\varepsilon\delta^{p-2}(x,x')\left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r^2 . \qquad (3.4)
\end{aligned}$$

Proof. The first inequality is a consequence of Lemma 3.2.1 in [1]. The second results from the algebraic inequality

$$\delta(x,x')\|z-z'\| \leq \frac{1}{\varepsilon}\delta^2(x,x') + \varepsilon\|z-z'\|^2,$$

the hypothesis $\Psi \approx \delta^p$, the equivalence of Riemannian and Euclidean norms, and the inequality (2.2). □

3.2 The case of a general connection

In this paragraph, the connection does not depend *a priori* on the Riemannian structure.

Lemma 3.2.1 *Let μ_0 be a positive real number. Each point q of the manifold M has a neighbourhood O_q^{int}, relatively compact in M and depending on the geometry of the manifold (but not on f), with the following property : if (X, Z) is a solution of equation $(M + D)$ such that $X \in \overline{O_q^{int}}$, then it belongs to $(\mathcal{E}_{\mu_0})$.*

Proof. We want to apply the result of Lemma 3.1.2; we consider the function $g(x) = \cos(a|x|)$ in a normal system of coordinates centered at q, where $a > 0$ and x is such that $|x| \leq r_0 < \pi/2a$ (a and r_0 are to be determined). Firstly, g is smooth and bounded, below and above, by positive constants.
Now, for $\mu > \mu_0$, we prove that we can find a large enough such that g verifies the inequality (3.1) with $\alpha = \mu$. Using the equivalence of the Riemannian and the Euclidean norms on the domain on which we are working (by shrinking it if necessary), we know that there is a $C_r > 0$ such that $|\cdot|_r^2 \leq C_r|\cdot|^2$. In particular, it is sufficient to show that for any $u \in \mathbb{R}^n$,

$$\text{Hess } g(x) < u, u > +2C_r\mu g(x)|u|^2 \leq 0. \qquad (3.5)$$

Recall that, for $u = (u_1, \ldots, u_n)$

$$\text{Hess } g(x) < u, u >= \sum_{i,j} \left(D_{ij}g(x) - \Gamma_{ij}^k(x)D_k g(x)\right) u_i u_j.$$

It is easy to compute the partial derivatives of g

$$\begin{aligned} D_k g(x) &= -a\sin(a|x|)\frac{x_k}{|x|} \\ D_{ij} g(x) &= -a^2\cos(a|x|)\frac{x_i x_j}{|x|^2} - a\sin(a|x|)\frac{\delta_{ij}}{|x|} + a\sin(a|x|)\frac{x_i x_j}{|x|^3} \end{aligned}$$

(where $\delta_{ij} = 1$ if $i = j$, 0 otherwise). So (3.5) is equivalent to showing that the following matrix

$$\left(-a^2 g(x)\frac{x_i x_j}{|x|^2} + a\sin(a|x|)\frac{x_i x_j}{|x|^3} + a\frac{\sin(a|x|)}{|x|}\left(\Gamma_{ij}^k(x)x_k - \delta_{ij}\right) + 2C_r\mu g(x)\delta_{ij}\right)_{ij} \tag{3.6}$$

is nonpositive (as a symmetric matrix).
We put $a = \sqrt{2\pi C_r\mu}$ and $k(x) = a\sin(a|x|)/|x|$; note that $k(x) \to a^2 > 0$ if $x \to 0$. Then

$$-\frac{1}{2}k(x) + 2C_r\mu g(x) = a^2\left(-\frac{\sin(a|x|)}{2a|x|} + \frac{1}{\pi}g(x)\right) \leq 0,$$

since $0 \leq a|x| < \pi/2$ and $\sin t \geq 2t/\pi$ for $t \in [0;\pi/2]$. Therefore the matrix

$$\left(-\frac{1}{2}k(x) + 2C_r\mu g(x)\right)(\delta_{ij})_{ij} \tag{3.7}$$

is nonpositive. Moreover we have

$$\begin{aligned} \left(-a^2 g(x)\frac{x_i x_j}{|x|^2} + a\sin(a|x|)\frac{x_i x_j}{|x|^3}\right)_{ij} &= a^4\left(\frac{-a|x|g(x) + \sin(a|x|)}{a^3|x|^3}\right)(x_i x_j)_{ij} \\ &\leq a^4(x_i x_j)_{ij} \end{aligned}$$

if $|x| \leq r_1$, for $(-t\cos t + \sin t)/t^3 \leq 1$ in a neighbourhood of 0 and the matrix $(x_i x_j)_{ij}$ is easily seen to be nonnegative. Thus the matrix

$$\left(-a^2 g(x)\frac{x_i x_j}{|x|^2} + a\sin(a|x|)\frac{x_i x_j}{|x|^3} - \frac{1}{4}k(x)\delta_{ij}\right)_{ij} \tag{3.8}$$

is nonpositive if $|x| \leq r_2 \leq r_1$. Now, for $|x| \leq r_3$, the matrix

$$\left(a\frac{\sin(a|x|)}{|x|}\left(\Gamma_{ij}^k(x)x_k - \delta_{ij}\right)\right)_{ij}$$

is less than $-(3/4)k(x)(\delta_{ij})_{ij}$. Using this and the nonpositivity of the matrices (3.7) and (3.8), we conclude that, for $|x| \leq r_0 = \inf(r_2, r_3, \pi/(4a))$, the matrix (3.6) is nonpositive.

This shows the inequality (3.5). As a consequence of Lemma 3.1.2, if we take $\overline{O_q^{int}} = \exp_q(\overline{B(0,r_0)})$, any solution (X, Z) of the equation $(M + D)$, with $X \in \overline{O_q^{int}}$, is in $(\mathcal{E}_{\mu-\varepsilon})$ for any $\varepsilon > 0$; it is in particular in $(\mathcal{E}_{\mu_0})$.
Remark that the drift f interferes in the proof (given in [1]) of Lemma 3.1.2, but the

result of this lemma is independent of f. Therefore the neighbourhood $\overline{O_q^{int}}$ does not depend on f. $\square$

The following theorem gives the uniqueness property in the case of a general connection, independent of the Riemannian structure.

Theorem 3.2.2 *Each point q of the manifold M has a neighbourhood O_q, relatively compact in M, and depending only on the geometry of the manifold (but not on the constants L and L_2 in the conditions* (1.3) *and* (1.4) *on f), with the following property : for a given terminal value U in O_q, there is at most one solution (X, Z) to the equation $(M + D)$ such that X remains in O_q.*

Proof. Fix $q \in M$; according to Lemma (4.59) of [4], q has a neighbourhood O_1, relatively compact in M, with Γ-convex geometry, with the convex function Ψ defined at the end of Subsection 2.3. We recall that this function Ψ verifies the estimate (2.6) near the diagonal (shrinking O_1 if necessary, we can suppose that it holds on $O_1 \times O_1$) and $\Psi \approx \delta^2$. Taking $p = 2$ in (3.4) and $\varepsilon = \alpha$ as in (2.6), we get

$$\begin{aligned} U_t &= \frac{1}{2}\sum_{i=1}^{d_w} {}^t[{}^t\tilde{Z}_t]^i \text{Hess } \Psi(\tilde{X}_t)[{}^t\tilde{Z}_t]^i + D\Psi(\tilde{X}_t)\begin{pmatrix} f(B_t^y, X_t, Z_t) \\ f(B_t^y, X_t', Z_t') \end{pmatrix} \\ &\geq -\frac{\beta}{2}\Psi(\tilde{X}_t)(\|Z_t\|_r^2 + \|Z_t'\|_r^2) - C_\alpha(1 + \|Z_t\|_r + \|Z_t'\|_r)\Psi(\tilde{X}_t). \end{aligned}$$

Thus for λ_0 and μ_0 large enough,

$$U_t + (\lambda_0 + \mu_0(\|Z_t\|_r^2 + \|Z_t'\|_r^2)\Psi(\tilde{X}_t) \geq 0.$$

Note that it suffices to take $\mu_0 > \beta/2$, and that β depends only on Ψ. Therefore μ_0 does not depend on the drift f. For these λ_0 and μ_0, it results from (3.2) that the process $(S_t)_t = (\exp(A_t)\Psi(\tilde{X}_t))_t$ is a submartingale under the additional integrability condition

$$\mathbb{E}\left(e^{\mu_0 \int_0^t (\|Z_s\|_r^2 + \|Z_s'\|_r^2)ds}\right) < \infty.$$

According to Lemma 3.2.1, this inequality holds if X remains in a small compact neighbourhood $\overline{O_q^{int}}$ of q. Thus the process $(S_t)_t$ is indeed a submartingale if X remains in $\overline{O_1 \cap O_q^{int}}$. Now the conclusion is classical : since the submartingale S is nonnegative and has terminal value 0, it vanishes identically. Therefore $\Psi(\tilde{X}_t) = 0$ for any t, and the definition of Ψ leads to $X_t = X_t'$. The proof is completed since we consider continuous processes. $\square$

3.3 Uniqueness for regular geodesic balls

In this subsection, the connection used is Levi-Civita's one. The way to get uniqueness is similar as the one in the preceding subsection; the difference is that we can obtain more precise results by achieving explicit calculations.

An accurate examination of formula (2.9) together with (3.3) shows that the μ (in (3.3)) needed to balance, near the diagonal, the second term in the Hessian is approximately $\frac{K}{4}$; this implies that the processes should be in $(\mathcal{E}_{\frac{K}{4}})$. As the next lemma shows, this condition requires that we work on smaller domains than in the nonpositive-curvature case treated in [1]. Let $o \in M$ and $\mathcal{B}_\rho$ denote the regular geodesic ball of radius ρ and centered at o.

Lemma 3.3.1 *Suppose that $0 < \gamma < 1$ is such that $\rho = \gamma \frac{\pi}{2\sqrt{K}}$; if we are given two solutions X, X' of $(M + D)$ remaining in $\mathcal{B}_\rho$, then for any $0 < e < 1/\gamma$,*

(i) X and X' belong to $(\mathcal{E}_{e\frac{K}{2}})$;

(ii) $\mathbb{E} \exp\left(e\frac{K}{4}\int_0^T (\|Z_s\|_r^2 + \|Z_s'\|_r^2)ds\right) < \infty.$

Proof. (i) Let $\varphi(x) = \cos(\beta\sqrt{K}\delta(o,x))$ with $1/\gamma > \beta > 1$. Then on $\mathcal{B}_\rho$, φ is smooth and there is $c > 0$ such that $1 \geq \varphi(x) \geq c > 0$. Moreover,

$$\begin{aligned}
\text{Hess } \varphi(x) < u, u > &= -K\beta^2\varphi(x)\left(\delta'(o,x) < u >\right)^2 \\
&\quad -\sqrt{K}\beta\sin(\sqrt{K}\beta\delta(o,x)) \cdot \text{Hess } \delta(o,x) < u, u > \\
&\leq -K\beta^2\varphi(x)|v|_r^2 \\
&\quad -K\beta\sin(\sqrt{K}\beta\delta(o,x))\cot(\sqrt{K}\delta(o,x))|w|_r^2.
\end{aligned}$$

The last inequality is a consequence of (2.8) and the estimation (1.2.2) of [10] :

$$\text{Hess } \delta(o,x) < u, u > \geq \sqrt{K}\cot(\sqrt{K}\delta(o,x))|w|_r^2. \tag{3.9}$$

As the function cotangent is decreasing on $]0; \frac{\pi}{2}]$ and $\beta > 1$, finally we get

$$\text{Hess } \varphi(x) < u, u > \leq -K\beta\varphi(x)|u|_r^2.$$

Now we can apply Lemma 3.1.2 : X and X' belong to $(\mathcal{E}_{K\beta/2} - \varepsilon)$ for any $\varepsilon > 0$; but since β is arbitrary between 1 and $1/\gamma$, X and X' belong to $(\mathcal{E}_{eK/2})$ for $0 < e < 1/\gamma$.

(ii) It is an immediate consequence of (i) and Cauchy-Schwarz inequality. □

From now on in this subsection, we consider only processes remaining in $\overline{\omega} = \mathcal{B}_\rho$ with $\rho = \gamma\frac{\pi}{2\sqrt{K}}$ $(0 < \gamma < 1)$ fixed. According to Lemma 3.3.1, fix $1 < e < 1/\gamma$ such that

$$\mathbb{E}\exp\left(e\frac{K}{4}\int_0^T (\|Z_s\|_r^2 + \|Z_s'\|_r^2)ds\right) < \infty;$$

then fix a such that $1 < a < 1 + (e-1)/2$. We are going to prove now that the sum (3.3) is nonnegative for $\Psi = \Psi_a$, the function defined in (2.10), and for suitable constants λ and μ.

Proposition 3.3.2 *Let $\Psi(\tilde{x}) = \sin^a\left(\sqrt{K}\delta(\tilde{x})/2\right)$ and σ, τ stopping times such that $0 \leq \sigma < \tau \leq T$. Then, if $\mathbb{P}[\forall t \in [\sigma;\tau], X_t \notin \Delta] = 1$, the sum (3.3) is nonnegative for $\sigma \leq t \leq \tau$ and $(S_t)_t = (\exp(A_t)\Psi(\tilde{X}_t))_t$ is a submartingale on the random interval $[\sigma;\tau]$, for λ large enough and $\mu = eK/4$ with e as above (λ and μ don't depend on σ nor τ).*

Proof. First remark that Ψ is smooth out of the diagonal but not on the diagonal, so we will consider in this proof only $\tilde{x} = (x, x') \notin \Delta$. Moreover, we put again $y := \sqrt{K}\delta(\tilde{x})/2$ and $h(y) := \sin y/y$. We complete the proof in two steps : first we work near the diagonal and then away from it.

Near the diagonal, we use part (i) of Lemma 2.4.3 with $\beta > 1$ such that $a\beta = 1 + (e-1)/2$ (this is possible since $a < 1 + (e-1)/2$). Let $b \in \mathbb{R}^d$, $x, x' \in (\overline{V}_\beta \setminus \Delta) \cap (\mathcal{B}_\rho \times \mathcal{B}_\rho)$, $z \in T_x M$, $z' \in T_{x'} M$ and $u = (z, z')$. On the one hand, we have from (2.11)

$$\text{Hess } \Psi(\tilde{x}) < u, u > \geq \alpha \sin^{a-2}(y) \left| \overset{x'}{\underset{x}{\|}} z - z' \right|_r^2 - \left(1 + \frac{e-1}{2}\right) \frac{K}{2} \Psi(\tilde{x}) (|z|_r^2 + |z'|_r^2).$$

On the other hand, for $z \in \mathcal{L}(\mathbb{R}^{d_w}, T_x M)$ and $z' \in \mathcal{L}(\mathbb{R}^{d_w}, T_{x'} M)$,

$$\begin{aligned}
\left| D\Psi(\tilde{x}) \cdot \begin{pmatrix} f(b, x, z) \\ f(b, x', z') \end{pmatrix} \right| &= a \sin^{a-1}(y) \cos y \cdot \frac{\sqrt{K}}{2} \left| \delta'(\tilde{x}) \cdot \begin{pmatrix} f(b, x, z) \\ f(b, x', z') \end{pmatrix} \right| \\
&\leq a \sin^{a-1}(y) \frac{\sqrt{K}}{2} \\
&\qquad \times L \left(\delta(\tilde{x})(1 + \|z\|_r + \|z'\|_r) + \left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r \right) \\
&\leq a \frac{\pi}{2} \Psi(\tilde{x}) L (1 + \|z\|_r + \|z'\|_r) + C_1 \sin^{a-1} y \left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r \\
&\leq C\Psi(\tilde{x}) + \frac{e-1}{2} \frac{K}{4} \Psi(\tilde{x}) (\|z\|_r^2 + \|z'\|_r^2) \\
&\qquad + \frac{\alpha}{2} \sin^{a-2} y \left\| \overset{x'}{\underset{x}{\|}} z - z' \right\|_r^2 .
\end{aligned}$$

The first inequality above is a consequence of (2.8) and (1.3), and the second one of the inequality $y \leq (\pi/2)\sin y$, because $0 < y < \pi/2$. The last one uses classical inequalities. Therefore, summing the terms Hess $\Psi(\tilde{X}_t) < [{}^t\tilde{Z}_t]^i, [{}^t\tilde{Z}_t]^i >$ for $i = 1, \ldots, d_w$, we get

$$\begin{aligned}
&\frac{1}{2} \text{Tr} \left({}^t\tilde{Z}_t \text{Hess } \Psi(\tilde{X}_t) \tilde{Z}_t \right) \\
&+ D\Psi(\tilde{X}_t) \begin{pmatrix} f(B_t^y, X_t, Z_t) \\ f(B_t^y, X'_t, Z'_t) \end{pmatrix} \geq -C\Psi(\tilde{X}_t) - e\frac{K}{4} \Psi(\tilde{X}_t) (\|Z_t\|_r^2 + \|Z'_t\|_r^2).
\end{aligned}$$

Taking $\lambda \geq C$ and $\mu = e\frac{K}{4}$, the sum (3.3) is nonnegative on $(\overline{V}_\beta \setminus \Delta) \cap (\mathcal{B}_\rho \times \mathcal{B}_\rho)$.

It remains to show the nonnegativity of (3.3) on $(\mathcal{B}_\rho \times \mathcal{B}_\rho) \setminus \overline{V}_\beta$. Using here estimation (2.12), we get, since $a < e$ (using the definition of a and e)

$$\begin{aligned}
\frac{1}{2} \text{Tr} \left({}^t\tilde{Z}_t \text{Hess } \Psi(\tilde{X}_t) \tilde{Z}_t \right) &+ \left(\lambda + e\frac{K}{4} (\|Z_t\|_r^2 + \|Z'_t\|_r^2)\right) \Psi(\tilde{X}_t) \\
&\geq \lambda \Psi(\tilde{X}_t) + (e-a) \frac{K}{4} \Psi(\tilde{X}_t) (\|Z_t\|_r^2 + \|Z'_t\|_r^2).
\end{aligned}$$

But we also have, for x, x' in $(\mathcal{B}_\rho \times \mathcal{B}_\rho) \setminus \overline{V}_\beta$,

$$\begin{aligned}
\left| D\Psi(\tilde{x}) \cdot \begin{pmatrix} f(b,x,z) \\ f(b,x',z') \end{pmatrix} \right| &\leq C_2(1 + \|z\|_r + \|z'\|_r) \\
&\leq C_3 + (e-a) \left(\min_{(\mathcal{B}_\rho \times \mathcal{B}_\rho)\setminus \overline{V}_\beta} \Psi \right) \frac{K}{4}(\|z\|_r^2 + \|z'\|_r^2).
\end{aligned}$$

Then taking $\lambda \geq C_3/(\min \Psi)$ gives the nonnegativity of the sum (3.3) outside $\overline{V}_\beta$. Then the sum (3.3) is always nonnegative, and the submartingale property follows. □

The next proposition extends the preceding result to processes that are allowed to live in the diagonal.

Proposition 3.3.3 *If $(\tilde{X}_t)_t$ remains in $\overline{\mathcal{B}}_\rho \times \overline{\mathcal{B}}_\rho$, then the process $(S_t)_{0 \leq t \leq T}$ is a submartingale.*

Proof. Let $S_t^\varepsilon := S_t \vee \varepsilon$ for $\varepsilon > 0$; we begin by proving that the process $(S_t^\varepsilon)_t$ is a submartingale. Indeed, let

$$\tau'_0 := 0, \ \tau_k := \inf\{t \geq \tau'_{k-1} : S_t \leq \frac{\varepsilon}{2}\} \ \text{ and } \ \tau'_k := \inf\{t \geq \tau_k : S_t \geq \varepsilon\}.$$

Then it is sufficient to show that for k fixed, (S_t^ε) is a submartingale on the random time intervals $[\tau_k; \tau'_k]$ and $[\tau'_k; \tau_{k+1}]$. Let σ and τ be stopping times; if $\tau_k \leq \sigma < \tau \leq \tau'_k$ and $A \in \mathcal{F}_\sigma$ then $\mathbb{E}\,(S_\tau^\varepsilon 1_A) = \mathbb{E}\,(S_\sigma^\varepsilon 1_A)$ since $S_\sigma^\varepsilon = S_\tau^\varepsilon = \varepsilon$. Now suppose that $\tau'_k \leq \sigma < \tau \leq \tau_{k+1}$ and $A \in \mathcal{F}_\sigma$; then $\mathbb{E}\,(S_\tau 1_A) \geq \mathbb{E}\,(S_\sigma 1_A)$ since (S_u) is a submartingale out of the diagonal. In particular,

$$\mathbb{E}\left(S_\tau^\varepsilon 1_A 1_{\{S_\sigma > \varepsilon\}}\right) \geq \mathbb{E}\left(S_\tau 1_A 1_{\{S_\sigma > \varepsilon\}}\right) \geq \mathbb{E}\left(S_\sigma 1_A 1_{\{S_\sigma > \varepsilon\}}\right) = \mathbb{E}\left(S_\sigma^\varepsilon 1_A 1_{\{S_\sigma > \varepsilon\}}\right);$$

moreover,

$$\mathbb{E}\left(S_\tau^\varepsilon 1_A 1_{\{S_\sigma \leq \varepsilon\}}\right) \geq \mathbb{E}\left(S_\sigma^\varepsilon 1_A 1_{\{S_\sigma \leq \varepsilon\}}\right).$$

So $\mathbb{E}\,(S_\tau^\varepsilon 1_A) \geq \mathbb{E}\,(S_\sigma^\varepsilon 1_A)$. Therefore, (S_t^ε) is indeed a submartingale. To conclude, it suffices to apply Lebesgue's dominated convergence theorem with $\varepsilon \to 0$, since $\sup_t S_t$ is integrable. □

The uniqueness property now follows as usual.

Theorem 3.3.4 *Suppose that the drift f verifies conditions* (1.3) *and* (1.4)*, and that $\overline{\omega} = \mathcal{B}_\rho$ is a regular geodesic ball. Then, for a given terminal value U in the compact $\overline{\omega}$, there is at most one $\overline{\omega}$-valued solution to the equation $(M + D)$.*

Remark : Theorem 3.3.4 is optimal in the following sense : if $\overline{\omega}$ is a geodesic ball of radius $r \geq \pi/(2\sqrt{K})$, then it is possible to exhibit two different martingales (in particular, two different solutions of the equation $(M + D)$ with $f = 0$) which have the same terminal value.

Let us consider the classical example of the sphere $\mathbb{S}^2$. We embed $\mathbb{S}^2$ in $\mathbb{R}^3$, as the sphere of radius 1. We call $N = (0,0,1)$ the northern pole and we take $\overline{\omega} = \mathcal{B}_{\rho_0}$, the geodesic ball centered at N of radius $\pi/(2\sqrt{K}) = \pi/2$. Note that $\overline{\omega}$ is nothing but the northern hemisphere, containing the equator

$$E = \{(x,y,z)/x^2 + y^2 = 1 \text{ and } z = 0\}.$$

Now let (B_t) be a BM starting at $(1,0,0)$, moving along the equator E and stopped when it reaches the plane $\{x = 0\}$. Define another BM (B'_t) moving along E by reflecting (B_t) with respect to the plane $\{x = 0\}$ (in particular, it starts from $(-1,0,0)$ and is stopped when it reaches the plane $\{x = 0\}$). These two processes are martingales on the sphere, since they are BM moving along a geodesic; moreover, it is obvious that they have the same terminal value (the point $(0,1,0)$ or $(0,-1,0)$). Hence, in this case, the uniqueness doesn't hold.
Note that this result can be extended to manifolds which have closed geodesics (see for instance Proposition 2.2.2 of [10]).

We conclude the uniqueness part by giving a consequence of the calculus achieved in the two preceding subsections, which will be useful in the proof of existence.

Proposition 3.3.5 *In the two cases above (general connection and regular geodesic balls), there is a $q_0 > 1$ such that the submartingale $(S_t)_t$ is in $L^q(\Omega)$ for $1 < q < q_0$.*

4 Existence

In this section we are given an $\overline{\omega}$-valued random variable U and we want to construct a pair of processes (X, Z), satisfying equation $(M + D)$, with X in $\overline{\omega}$ and terminal value U. We limit ourselves to the case of a Wiener probability space. **This part is so similar as the existence part of [1] that we just give the main results and the changes to make in the proofs.**

We recall the strategy of the proof in [1] :
1. Simplify the problem by considering only terminal values which can be expressed as functions of the diffusion B^y at time T, i.e. $U = F(B^y_T)$. This step needs to pass through the limit in equation $(M + D)$.
2. Solve a Pardoux-Peng BSDE with parameter to construct a pair of processes in $\mathbb{R}^n \times \mathbb{R}^{nd_w}$ which is close to being a solution of $(M + D)$ with $X_T = U$.
3. Show that under an additional condition on f the solution of the preceding BSDE is a solution of the BSDE $(M + D)$ on a small time interval.
4. Use the convex function Ψ to show that we have a solution of $(M + D)$ on the whole time interval $[0; T]$.
In fact, for technical reasons we suppose in the two last steps that f is sufficiently regular; then the proof of the existence is completed with the last subsection :
5. Solve BSDE $(M + D)$ for general f using classical approximation methods.

Note that we usually work within local coordinates in $\mathbb{R}^n$, i.e. we consider that $\overline{\omega} \subset O \subset \mathbb{R}^n$.

4.1 Reduction of the problem

For an $\overline{\omega}$-valued random variable U^l, we denote by (X^l, Z^l) the solution of BSDE $(M+D)$ with terminal value U^l, and such that X^l remains in $\overline{\omega}$.

Proposition 4.1.1 *If U^l tends to U in $L^2(\Omega)$ as $l \to \infty$, then the processes X^l tend to an $\overline{\omega}$-valued process X for the square norm $\mathbb{E}(\sup_{t\in[0,T]} |X_t^l - X_t|^2)$, and the processes Z^l to a process Z for the square norm*

$$\mathbb{E}\left(\int_0^T \|Z_t^l - Z_t\|^2 dt\right).$$

Moreover, (X, Z) is the $\overline{\omega}$-valued solution of BSDE $(M+D)$ with terminal value U.

Proof. For integers l, m, we put

$$\tilde{X}^{l,m} = (X^l, X^m) \quad \text{and} \quad \tilde{Z}^{l,m} = \begin{pmatrix} Z^l \\ Z^m \end{pmatrix}.$$

We first deal with the existence of X. An application of Doob's L^p inequality to the submartingale $(S_t)_t = (\exp(A_t)\Psi(\tilde{X}_t))_t$ $(S_t)_t$, with $1 < q < q_0$ as in Proposition 3.3.5, shows that, since $(U^l)_l$ is Cauchy, the sequence $(X^l)_l$ is Cauchy for the sup norm

$$\delta^{(2)}\left((X_t^l), (X_t)\right) = \sqrt{\mathbb{E}\left(\sup_{t\in[0,T]} \delta^2(X_t^l, X_t)\right)}.$$

The proof is similar to the one of Lemma 4.1.2 in [1], but we recall briefly the method for the convenience of the reader. Since $\Psi \approx \delta^p$, we have

$$\begin{aligned} \mathbb{E}\left(\sup_t \delta^p(X_t^l, X_t^m)\right) &\le C\mathbb{E}\left(\sup_t \left(e^{A_t}\Psi(X_t^l, X_t^m)\right)^q\right)^{\frac{1}{q}} \\ &\le C\mathbb{E}\left(e^{qA_T}\Psi^q(X_T^l, X_T^m)\right)^{\frac{1}{q}} \\ &\le C\mathbb{E}\left(e^{q\tilde{q}A_T}\right)^{\frac{1}{q\tilde{q}}} \mathbb{E}\left(\delta^{\frac{q\tilde{q}p}{\tilde{q}-1}}(U^l, U^m)\right)^{\frac{\tilde{q}-1}{q\tilde{q}}} \end{aligned} \tag{4.1}$$

The constant C above is allowed to vary from one inequality to another, but it depends only on $T, \overline{\omega}$ and Ψ (but not on the processes (X^l)). The second inequality is Doob's L^q one applied to the submartingale $(\exp(A_t)\Psi(X_t, X_t'))_t$; and the third one is Hölder's one with $\tilde{q}$.
We choose q and $\tilde{q}$ such that $q, \tilde{q} > 1$ and $q\tilde{q} < q_0$ with q_0 as in Proposition 3.3.5, so that $\exp(q\tilde{q}A_T)$ is integrable. Note that $\mathbb{E}(\exp(q\tilde{q}A_T))$ is uniformly bounded by Lemmas 3.2.1 and 3.3.1. Moreover, there are positive constants $\tilde{c}$, γ_1 and γ_2 such that, for any $\gamma > 1$ and any variables X_1 and X_2 in $\overline{\omega}$,

$$\tilde{c}\mathbb{E}(\delta^\gamma(X_1, X_2))^{\frac{1}{\gamma_1}} \le \mathbb{E}(\delta^2(X_1, X_2)) \le \frac{1}{\tilde{c}}\mathbb{E}(\delta^\gamma(X_1, X_2))^{\frac{1}{\gamma_2}};$$

this easily results from the boundedness of δ or Hölder's inequality. At the end we get, for a constant $\zeta > 0$

$$\delta^{(2)}\left((X_t^l),(X_t)\right) \leq C\mathbb{E}\left(\delta^2(U^l,U^m)\right)^{\zeta}.$$

Now if $(U^l)_l$ is Cauchy in L^2, the sequence $(X^l)_l$ is clearly Cauchy for the sup norm. Thus $(X^l)_l$ converges to a process X.

Then we seek a process Z. We consider here the function $\hat{\Psi}$ defined by :

$$\begin{cases} \hat{\Psi} = \Psi \approx \delta^2 & \text{if a general connection is used (see (2.7));} \\ \hat{\Psi} = \delta^2 & \text{in the case of the Levi-Civita connection.} \end{cases}$$

For this function $\hat{\Psi}$, there is $a > 0$ and $b > 0$ such that

$$\forall (x,x') \in \overline{\omega} \times \overline{\omega},\ \forall (z,z') \in T_xM \times T_{x'}M,$$

$$^t\begin{pmatrix} z \\ z' \end{pmatrix} \text{Hess } \hat{\Psi}(x,x') \begin{pmatrix} z \\ z' \end{pmatrix} \geq \alpha \left| \overset{x'}{\underset{x}{\|}} z - z' \right|_r^2 - \beta \hat{\Psi}(x,x')(|z|_r^2 + |z'|_r^2). \tag{4.2}$$

In the case of a general connection, this is a consequence of (2.6). If the Levi-Civita connection is used, (4.2) results from (2.8) and (2.9), remarking that, for $u \in T_xM \times T_{x'}M$,

$$\text{Hess } \delta^2(x,x') < u,u >= 2\left(\delta(x,x')\text{Hess } \delta(x,x') < u,u > + (\delta'(x,x') < u >)^2\right).$$

Now apply Itô's formula (2.1) to $\hat{\Psi}(\tilde{X}^{l,m})$ and use estimates (4.2) and (3.4) with $\varepsilon = \alpha/2$ to write

$$\begin{aligned} \frac{\alpha}{2}\mathbb{E}\int_0^T \left\| \overset{X_s^l}{\underset{X_s^m}{\|}} Z_s^m - Z_s^l \right\|_r^2 ds \quad &\leq \quad \mathbb{E}\left(\hat{\Psi}(\tilde{X}_T^{l,m}) + \hat{\Psi}(\tilde{X}_0^{l,m})\right) \\ &+ C_\varepsilon \mathbb{E}\int_0^T \hat{\Psi}(\tilde{X}_s^{l,m})(1 + \|Z_s^l\|_r + \|Z_s^m\|_r)ds \\ &+ \beta\mathbb{E}\int_0^T \hat{\Psi}(\tilde{X}_s^{l,m})(\|Z_s^l\|_r^2 + \|Z_s^m\|_r^2)ds. \end{aligned}$$

Since $(X_l)_l$ is Cauchy for the sup L^2-norm, the three expectations on the right tend to zero as $l,m \to \infty$. Indeed, for the first expectation, it suffices to note that $\hat{\Psi} \approx \delta^2$, and for the last two, to use Hölder's inequality and the fact that

$$\int_0^T (\|Z_s\|_r^2 + \|Z_s'\|_r^2)ds$$

has some uniformly bounded exponential moments (from Lemmas 3.2.1 and 3.3.1). An application of the inequality (2.2) and an argument similar to that used just above show that, for the Euclidean norm,

$$\mathbb{E}\int_0^T \|Z_s^m - Z_s^l\|^2 ds \overset{l,m\to\infty}{\longrightarrow} 0.$$

Then $(Z^l)_l$ is a Cauchy sequence for the square norm $\mathbb{E}(\int_0^T \|Z_t^l - Z_t^m\|^2 dt)$ and by completeness, there is a limit process Z.

To complete the proof, it remains, by passing through the limit, to show that (X, Z) is the solution of BSDE $(M + D)$ with $X_T = U$. This is easy and has yet been completed in the Second Step of the proof of Proposition 4.1.4 in [1]. □

Using the density of the space of all functionals

$$\{G(W_{t_1}, W_{t_2}, \ldots, W_{t_q}), \quad 0 < t_1 < \ldots < t_q \leq T; \\ G : \mathbb{R}^{qd_W} \to \overline{\omega} \text{ smooth, constant off a compact set } \}$$

in $L^2(\mathcal{F}_T; \overline{\omega})$ and an argument of conditioning (see the end of Section 4.1 in [1]), it turns out that it suffices to solve equation $(M + D)$ with a terminal value U that can be written $F(B_T^y)$ for smooth $F : \mathbb{R}^d \to \overline{\omega}$, constant off a compact set.

4.2 The solution for terminal values $U = F(B_T^y)$

The proof of the existence of such a solution corresponds to the steps 2, 3 and 4 given at the beginning of Section 4. In [1], they are dealt with in Subsections 4.2, 4.3, 4.4, 4.5. An accurate examination of these subsections shows that the proofs go essentially the same for the two cases here. The only change to make is in the proof of Proposition 4.5.1, when we apply Doob's inequality to the submartingale $(S_t)_t$: instead of an L^2 inequality, we have to consider an L^p inequality for p near 1, as in Proposition 3.3.5. Therefore, under the additional condition

(H_s) f is pointing strictly outward on the boundary $\partial\overline{\omega}$ of $\overline{\omega}$,

which means that (for the Riemannian inner product $(\cdot|\cdot)_r$)

$$\forall (b, x, z) \ : \ x \in \partial\overline{\omega}, \quad \inf_{b,x,z} (D\chi(x)|f(b, x, z))_r \geq \zeta > 0, \tag{4.3}$$

we have the following existence result.

Proposition 4.2.1 *Consider the BSDE* $(M+D)$ *with a terminal value* $U = F(B_T^y)$ *in* $\overline{\omega} = \{\chi \leq c\}$. *Suppose that* f *is a* C^3 *function with bounded derivatives, which verifies conditions* (1.3), (1.4) *and* (H_s); *suppose moreover that* χ *is strictly convex (i.e.* Hess χ *is positive definite). Then*

(i) If a general connection is used, there is a solution (X, Z) *to* $(M+D)$ *with* $X \in \overline{\omega}$.

(ii) If $\overline{\omega} \subset \mathcal{B}_\rho$ *(a regular geodesic ball of radius* ρ *and center* o*), there is again a solution* (X, Z) *to* $(M + D)$ *with* $X \in \overline{\omega}$.

Remark : If $\overline{\omega} = \mathcal{B}_\rho$ in (ii), the condition on χ is fulfilled. Indeed, $\mathcal{B}_\rho = \{\chi \leq c\}$ for a c such that $0 \leq c \leq \pi^2/(4K)$ and the smooth convex function (on $\mathcal{B}_\rho$) $\chi(x) = \delta^2(o, x)$. The strict convexity of χ on $\mathcal{B}_\rho$ comes from (2.8), (3.9) and the classical formula, for $z \in T_x M$,

$$\text{Hess } \chi(o, x) < z, z >= 2 \left(\delta(o, x)\text{Hess } \delta(o, x) < z, z > +(\delta'(o, x) < z >)^2\right) :$$

these formulas imply that, for $z = v + w \in T_xM$ (where v and w are the tangential and orthogonal components of z (see Subsection 2.4)),

$$\begin{aligned} \mathrm{Hess}\ \chi(o,x) < z, z > &\geq \sqrt{K}\delta(o,x)\cot(\sqrt{K}\delta(o,x))|w|_r^2 + |v|_r^2 \\ &\geq \eta|z|_r^2 \ \text{ on } \mathcal{B}_\rho, \text{ for } \eta > 0. \end{aligned}$$

In this case, the condition (4.3) is equivalent to

$$\forall (b,x,z)\ :\ x \in \partial\overline{\omega},\quad \inf_{b,x,z} \delta'(o,x) < f(b,x,z) > \geq \zeta > 0.$$

To get the existence in a general framework, it remains to extend Proposition 4.2.1 to general terminal values U and nonregular f, verifying the weaker condition

$$(H)\ \ f \text{ is pointing outward on the boundary of } \overline{\omega}.$$

Now it means that $\forall (b,x,z)\ :\ x \in \partial\overline{\omega},\ \ (D\chi(x)|f(b,x,z))_r \geq 0$. Note that this is a natural condition since it is necessary in the deterministic case (i.e. when the terminal value U is deterministic and $Z = 0$); moreover, in the case of a regular geodesic ball, it is equivalent to require that $\delta'(o,x) < f(b,x,z) >$ be only nonnegative. This generalization follows from Proposition 4.1.1, and Subsection 4.6 of [1]. At the end, we get the existence result of Theorem 1.3.1.

5 Applications

5.1 The martingale case

In the case of a vanishing drift f, solving equation $(M + D)$ is equivalent to finding a martingale on M with terminal value U. Then we recover Kendall's results for regular geodesic balls, stated in [6].

5.2 Case of a random terminal time

This case is the same as in [1]. So we just give the results obtained. The equation which we now study is

$$(M+D)_\tau \begin{cases} dX_t = Z_t dW_t + \left(-\frac{1}{2}\Gamma_{jk}(X_t)([Z_t]^k|[Z_t]^j) + f(B_t^y, X_t, Z_t)\right) dt \\ X_\tau = U^\tau \end{cases}$$

where U^τ is a $\overline{\omega}$-valued, $\mathcal{F}_\tau$-measurable random variable and τ is a stopping time which verifies the exponential integrability condition

$$\exists \xi > 0 : \mathbb{E}(e^{\xi\tau}) < \infty. \tag{5.1}$$

In this case, we need an additional restriction on f to keep the integrability condition

$$\mathbb{E}\left(e^{\lambda\tau + \mu\int_0^\tau (\|Z_s\|_r^2 + \|Z_s'\|_r^2) ds}\right) < \infty. \tag{5.2}$$

The condition is that f should be "small", in the following sense : there is an $h < \xi$ such that if the Lipschitz constant L and the bound L_2 (respectively in (1.3) and (1.4)) of f are smaller than h, then the integrability condition in (5.2) holds (in particular, it implies that $\lambda < \xi$). In this case, $(S_t)_{0 \le t \le \tau}$ remains a submartingale, and the consequence is the

Theorem 5.2.1 *We consider the BSDE $(M+D)_\tau$ with τ a stopping time verifying the integrability condition (5.1); the function χ used to define the domain $\overline{\omega}$ is supposed as usual to be strictly convex. Then if f verifies conditions (1.3), (1.4), (H) and moreover is "small" (in the sense defined above), this BSDE has a unique solution (X, Z), in the same cases as in Theorem 1.3.1.*

Note that if the stopping time τ is bounded a.s., then (5.2) holds (and therefore the existence and uniqueness of a solution) without supposing that f should be "small".

We end this paper by recalling briefly from [1] the applications to PDEs. The reader is referred to paragraphs 5.4 and 5.5 of [1] for further details.

5.3 Application to nonlinear PDEs

Under some conditions on the coefficients σ and b in the definition (1.1) of the diffusion B^x, we can think of this diffusion as a Brownian Motion on a Riemannian manifold (N, g). Let $\overline{M}_1$ be a compact submanifold of N, with boundary ∂M_1 and interior M_1. Given a regular mapping $\overline{\phi} : \partial M_1 \to \overline{\omega} \subset M$, we wish to find a mapping $\phi : \overline{M}_1 \to \overline{\omega}$ which solves the Dirichlet problem

$$(D) \left\{ \begin{array}{ccc} \mathcal{L}_M \phi(x) - f(x, \phi(x), \nabla\phi(x)\sigma(x)) = 0 & , & x \in M_1 \\ \phi(x) = \overline{\phi}(x) & , & x \in \partial M_1 \end{array} \right.$$

where $\mathcal{L}_M \phi$ is the tension field of the mapping ϕ.

Let ζ denote the first time B^x hits the boundary; we assume that ζ verifies an integrability condition like (5.1). Using the same Wiener process W with which we constructed B^x, we can solve according to Theorem 5.2.1 the BSDE $(M+D)_\zeta$ with terminal value $\overline{\phi}(B^x_\zeta)$. Let $(X^x_t, Z^x_t)_{0 \le t \le \zeta}$ be the unique solution and put $\phi(x) := X^x_0$. Then under sufficient regularity on ϕ, it is not difficult to verify that ϕ is a solution to the Dirichlet problem (D).

Note also that when $f(b, x, z) = f(b, x)$ and is written as $f(b, x) = D_2 G(b, x)$ (the differential of G with respect to the second variable), the elliptic nonlinear PDE in the Dirichlet problem (D) is associated with a variational problem; but it seems harder to associate variational problems for more general f.

We conclude by remarking that we can solve, as in [1], the heat equation associated with the elliptic problem (D).

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